Available online at www.sciencedirect.com
ScienceDirect
Journal of Pure and Applied Algebra 211 (2007) 773–783
JOURNAL OF PURE AND APPLIED ALGEBRA
www.elsevier.com/locate/jpaa

Gorenstein injective, Gorenstein flat modules and the section functor

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Received 6 December 2006; received in revised form 15 March 2007; accepted 5 April 2007
Available online 2 June 2007

Communicated by A.V. Geramita

Abstract

Let R be a commutative Noetherian ring of Krull dimension d, and let $\mathfrak{a}$ be an ideal of R. In this paper, we will study the strong cotorsioness and the Gorenstein injectivity of the section functor $\Gamma_{\mathfrak{a}}(-)$ in local cohomology. As applications, we will find new characterizations for Gorenstein and regular local rings. We also study the effect of the section functors $\Gamma_{\mathfrak{a}}(-)$ and the functors $H^d_{\mathfrak{a}}(-)$ on the Auslander and Bass classes.

MSC: Primary: 13D45; 13D07; 13H05; 13H10

1. Introduction

Throughout this paper, R is a commutative Noetherian ring (with identity). Grothendick local cohomology theory is an effective tool for mathematicians working in the theory of commutative algebras and in algebraic geometry. For example, it plays a substantial role in computing the depth and dimension of finitely generated modules over a local ring. There are several ways to compute these cohomology modules. One way is by the use of the right derived functors of the $\mathfrak{a}$-torsion functor $\Gamma_{\mathfrak{a}}(-) = \bigcup_{n\in\mathbb{N}}(0 :_{(-)} \mathfrak{a}^n)$, where $\mathfrak{a}$ is an arbitrary ideal of R. The functor $\Gamma_{\mathfrak{a}}(-)$ has its origins in algebraic geometry and it is called the *section functor*. In a different way, the local cohomolgy modules can be also computed by using Ext_R. More specifically, the *ith local cohomology of the module N with respect to the ideal* $\mathfrak{a}$, denoted $H^i_{\mathfrak{a}}(N)$, is isomorphic to $\varinjlim_n \mathrm{Ext}^i_R(R/\mathfrak{a}^n, N)$. When one applies local cohomology to commutative algebra problems, without loss of generality, one may assume that $\Gamma_{\mathfrak{a}}(N) = 0$. This condition plays a substantial role in the proof. The assertion of this condition depends on the basic fact that the section functor $\Gamma_{\mathfrak{a}}(I)$ is injective for any injective R-module I. Our main aim in this paper is to study the effect of the section functors on some classes of modules whose objects have a close relation with injective modules.

Now we briefly give some details of our results. Section 2 contains preliminaries, and in this section we shall state some definitions which we use later.

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0022-4049/$ - see front matter
doi:10.1016/j.jpaa.2007.04.013

Since generalized local cohomology modules are useful tools in our investigations, in Section 3 we present some results about Gorenstein injective modules and generalized local cohomology. Let $\mathfrak{a}$ be any ideal of R and M be any finitely generated R-module. We show that, for any G-injective module G and any non-negative integer i, there is an isomorphism $H^i_{\mathfrak{a}}(M, G) \cong \mathrm{Ext}^i_R(M, \Gamma_{\mathfrak{a}}(G))$ (see Proposition 3.1). Moreover for any R-module N of finite G-injective dimension with $\mathrm{Gid}_R N = t$ and any $j > t$, there is an isomorphism $H^j_{\mathfrak{a}}(M, N) \cong \mathrm{Ext}^{j-t}_R(M, \Gamma_{\mathfrak{a}}(\Omega^t(N)))$ where $\Omega^t(N)$ is the t-th cosyzygy of N (see Proposition 3.2). A similar fact holds for strongly cotorsion modules when R is Cohen–Macaulay and $\mathfrak{a}$ is maximal (see Proposition 3.4). We also show that if R is any ring of Krull dimension d and G is a G-injective R-module, then $\Gamma_{\mathfrak{a}}(G)$ is strongly cotorsion for any ideal $\mathfrak{a}$ of R. This result allows us to compute generalized local cohomology by using an arbitrary G-injective resolution of the second component (see Theorem 3.7). Finally in terms of the above result, we find new characterizations of Gorenstein and regular local rings (see Theorems 3.8 and 3.9).

As Foxby classes have a close relation with the classes of injective modules, in Section 3 we focus our studies on the effect of the section functor on these classes of modules. The Foxby classes contain two classes of R-modules, the so-called *Auslander* and *Bass classes*. The Bass class is a kind of dual to the Auslander class. In Section 3, we study the effect of the section functor $\Gamma_{\mathfrak{a}}(-)$ on the Bass class. We recall that $\Gamma_{\mathfrak{a}}(-)$ is a left exact functor, and if R is of Krull dimension d, then $H^d_{\mathfrak{a}}(-)$ is a right exact functor. So it seems that the study of the effect of the functor $H^d_{\mathfrak{a}}(-)$ on the Auslander class is also of interest. Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring admitting a dualizing module ω. In this section, we show that if $\mathfrak{a}$ is an ideal of R such that $H^1_{\mathfrak{a}}(\omega, G) = 0$ for any G-injective module G, then $\Gamma_{\mathfrak{a}}(G)$ is G-injective for any G-injective module G (see Theorem 4.7). As a result of this theorem, if $\mathfrak{a}$ is generated by an R-sequence or an ω-sequence, then $\Gamma_{\mathfrak{a}}(G)$ is G-injective for any G-injective module G. In particular, the conclusion holds for $\mathfrak{a} = \mathfrak{m}$. Let $\mathcal{J}_0(R)$ be the Bass class. We show that if N is a strongly cotorsion R-module such that $N \in \mathcal{J}_0(R)$, then $\Gamma_{\mathfrak{m}}(N) \in \mathcal{J}_0(R)$. Let $\mathcal{G}_0(R)$ be the Auslander class. We prove that if M is G-flat, then $H^d_{\mathfrak{m}}(M) \in \mathcal{G}_0(R)$ (see Theorem 4.12). We also prove that if M is a strongly torsion free R-module such that $M \in \mathcal{G}_0(R)$, then $H^d_{\mathfrak{m}}(M) \in \mathcal{G}_0(R)$. And finally, we show that if M is a maximal Cohen–Macaulay R-module such that $M \in \mathcal{G}_0(R)$. Then $\mathrm{Hom}_R(M, \omega) \in \mathcal{J}_0(R)$ (see Theorem 4.14).

2. Preliminaries

In this section, we recall some definitions that we use later.

Definition 2.1. Xu [14, Definitions 5.4.2 and 5.4.1] has introduced the notion of a strongly torsion free and of a strongly cotorsion module. An R-module M is said to be *strongly torsion free (strongly cotorsion)* if $\mathrm{Tor}^R_1(F, M) = 0$ ($\mathrm{Ext}^1_R(F, M) = 0$) for any R-module F of finite flat dimension. One can easily show that M is strongly torsion free (strongly cotorsion) if $\mathrm{Tor}^R_i(F, M) = 0$ ($\mathrm{Ext}^i_R(F, M) = 0$) for any $i \geq 1$ and any R-module F of finite flat dimension.

Definition 2.2. Following [5], an R-module N is said to be *Gorenstein injective (or G-injective)* if there exists a $\mathrm{Hom}(\mathcal{I}, -)$ exact exact sequence

$$\cdots \to E_1 \to E_0 \to E^0 \to E^1 \cdots$$

of injective R-modules such that $N = \mathrm{Ker}(E^0 \to E^1)$. Dually, an R-module M is said to be *Gorenstein flat (or G-flat)* if there exists an $\mathcal{I} \bigotimes -$ exact exact sequence

$$\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

of flat R-modules such that $M = \mathrm{Ker}(F^0 \to F^1)$.

It should be noted that if R is a Gorenstein ring of Krull dimension d (or a d-Gorenstein ring), then the strongly cotorsion (strongly torsion free) modules are precisely the G-injective (G-flat) modules.

We can define the *G-injective dimension* of an R-module N, $\mathrm{Gid}_R N$ as follows.

$\mathrm{Gid}_R N = \inf\{n \in \mathbb{N}_0 |$ there exists a $\mathcal{GI}-$ resolution $N \to \mathbf{E}$ of length $\leq n\}$, where $\mathcal{GI}$ denotes the class of all G-injective modules. We also denote by $\widetilde{\mathcal{I}}(\widetilde{\mathcal{GI}})$ the class of all modules of finite injective dimension (finite G-injective dimension). Dually, we can define the *G-flat dimension* of an R-module M, $\mathrm{Gfd}_R M$ as follows.

$\mathrm{Gfd}_R M = \inf\{n \in \mathbb{N}_0 |$ there exists a $\mathcal{GF}-$ resolution $\mathbf{F} \to M$ of length $\leq n\}$, where $\mathcal{GF}$ denotes the class of all G-flat modules and a $\mathcal{GF}-$resolution $\mathbf{F} \to M$ of length $\leq n$ is an exact sequence $0 \to F_m \to F_{m-1} \to \cdots \to F_0 \to M \to 0$ of R-modules such that each F_i is G-flat and $m \leq n$. We also denote by $\widetilde{\mathcal{F}}$ ($\widetilde{\mathcal{GF}}$) the class of all modules of finite flat dimension (finite G-flat dimension).

Definition 2.3. Following [4], an R-module M is said to be *copure flat* (*copure injective*) if $\mathrm{Tor}_1^R(E, M) = 0$ ($\mathrm{Ext}_R^1(E, M) = 0$) for any injective R-module E. Also, M is said to be *strongly copure flat (strongly copure injective)* if $\mathrm{Tor}_i^R(E, M) = 0$ ($\mathrm{Ext}_R^i(E, M) = 0$) for any injective R-module E and any $i > 0$. Note that in the definitions of strongly copure flat and strongly copure injective, the injectivity condition of module E can be replaced by a finite injective dimension. We say that an R-module N has *copure injective dimension* at most r, denoted $\mathrm{cid}_R N \leq r$, if there is an exact sequence $0 \to N \to C^0 \to C^1 \to \cdots \to C^r \to 0$ of R-modules with each C^i strongly copure injective. If there is no shorter such sequence, we set $\mathrm{cid}_R N = r$. Dually, we say that an R-module M has *copure flat dimension* at most r, denoted $\mathrm{cfd}_R M \leq r$, if there is an exact sequence $0 \to F_r \to \cdots \to F_1 \to F_0 \to M \to 0$ of R-modules with each F_i strongly copure flat. If there is no shorter such sequence, we set $\mathrm{cfd}_R M = r$.

Definition 2.4. Let $\mathfrak{a}$ be any ideal of R, $i \in \mathbb{N}_0$ and M and N be any R-module. By the definition of Herzog, the *ith generalized local cohomology of M and N with respect to* $\mathfrak{a}$, is denoted by $H_{\mathfrak{a}}^i(M, N)$, which is isomorphic to $\varinjlim_n \mathrm{Ext}_R^i(M/\mathfrak{a}^n M, N)$.

It should be noted that, in the definition above, if M is finitely generated and $N \to \mathbf{E}$ is an injective resolution of N, then the ith generalized local cohomology module of M and N with respect to $\mathfrak{a}$ is the ith cohomology module of the complex $\mathrm{Hom}_R(M, \Gamma_{\mathfrak{a}}(\mathbf{E}))$.

3. Strongly cotorion and Gorenstein injective modules

It follows from the proof of [11, Theorem 3.1] that if $\mathfrak{a}$ is an ideal of a commutative Noethrian ring R (without any other conditions on R) and G is a G-injective R-module, then $H_{\mathfrak{a}}^i(G) = 0$ for all $i > 0$. This fact gives us the following proposition.

Proposition 3.1. *Let $\mathfrak{a}$ be an ideal of R and let M be a finitely generated R-module. Let G be a G-injective R-module. Then $H_{\mathfrak{a}}^i(M, G) \cong \mathrm{Ext}_R^i(M, \Gamma_{\mathfrak{a}}(G))$ for all $i \in \mathbb{N}_0$. In particular, if G is $\mathfrak{a}$-torsion free, then $H_{\mathfrak{a}}^i(M, G) = 0$ for all $i \in \mathbb{N}_0$.*

Proof. As G is G-injective, there exists an exact sequence

$$0 \to G \to E^0 \to E^1 \to E^2 \to \cdots$$

of R-modules such that each of E^i is injective. Applying the functor $\Gamma_{\mathfrak{a}}(-)$ to this exact sequence and using [11, Theorem 3.1], we have the following exact sequence

$$0 \to \Gamma_{\mathfrak{a}}(G) \to \Gamma_{\mathfrak{a}}(E^0) \to \Gamma_{\mathfrak{a}}(E^1) \to \Gamma_{\mathfrak{a}}(E^2) \to \cdots.$$

By basic properties of the section functor $\Gamma_{\mathfrak{a}}(-)$, this exact sequence is an injective resolution for $\Gamma_{\mathfrak{a}}(G)$. Now, if we apply the functor $\mathrm{Hom}_R(M, -)$ to this exact sequence, we get our result. The second claim follows easily from the first part. □

Proposition 3.2. *Let $\mathfrak{a}$ be an ideal of R, and let M be a finitely generated R-module. Assume that N is an R-module with $\mathrm{Gid}_R N = t$. Then for all $j > 0$, we have $H_{\mathfrak{a}}^{t+j}(M, N) \cong \mathrm{Ext}_R^j(M, \Gamma_{\mathfrak{a}}(\Omega^t(N)))$, where $\Omega^t(N)$ is t-th cosyzygy of an injective resolution of N.*

Proof. To show the result, let the exact sequence $0 \to N \to \mathbf{E}$ be an injective resolution of N in which $\mathbf{E} = 0 \to E^0 \to E^1 \to \cdots \to E^{t-1} \to E^t \to E^{t+1} \to \cdots$. Let $\Omega^i(N) = \mathrm{Ker}(E^i \to E^{i+1})$ be i-th cosyzygy module of N. Since $\mathrm{Gid}_R N = t$, the module $\Omega^i(N)$ is G-injective for each $i \geq t$ by [6, Lemma 4.2]. One can easily show that $H_{\mathfrak{a}}^{t+j}(M, N) \cong H_{\mathfrak{a}}^j(M, \Omega^t(N))$. Now, the result follows by Proposition 3.1. □

Definition 3.3. Following [12], an R-module M is said to be *weakly cotorsion* if $\mathrm{Ext}^i_R(F, M) =$ for all finitely generated R-modules F of finite flat dimension.

A conclusion similar to Proposition 3.1 holds for strongly cotorsion modules when $(R, \mathfrak{m})$ is a Cohen–Macaulay local ring and $\mathfrak{a} = \mathfrak{m}$.

Proposition 3.4. *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring, let M be a finitely generated R-module, and let S be a strongly cotorsion R-module. Then $H^i_{\mathfrak{m}}(M, S) \cong \mathrm{Ext}^i_R(M, \Gamma_{\mathfrak{m}}(S))$ for all $i \in \mathbb{N}_0$. In particular, if S is $\mathfrak{m}$-torsion free or M is of finite projective dimension, then $H^i_{\mathfrak{m}}(M, S) = 0$ for all $i > 0$.*

Proof. As R is Cohen–Macaulay, it follows from [12, Theorem 2.2] that $H^i_{\mathfrak{m}}(S) = 0$ for all $i > 0$. Now the proof is similar to the proof of Proposition 3.1. In the second part, when S is $\mathfrak{m}$-torsion free, the result is trivial and when M is of finite projective dimension, according to [12, Proposition 2.16], $\Gamma_{\mathfrak{m}}(S)$ is weakly cotorsion; and hence the assertion follows by the first part. $\square$

In [11], the author proved that if R is d-Gorenstein and M is a G-injective R-module, then $\Gamma_{\mathfrak{a}}(M)$ is G-injective for any ideal of R. In the following proposition, we give a more general version of this theorem by removing the Gorenstein condition on R.

Proposition 3.5. *Let R be a ring of Krull dimension d, and let M be a G-injective R-module. Then $\Gamma_{\mathfrak{a}}(M)$ is strongly cotorsion for any ideal $\mathfrak{a}$ of R.*

Proof. As M is G-injective, there exists an exact sequence

$$\cdots \to E_2 \to E_1 \to E_0 \to M \to 0$$

of R-modules such that each E_i is injective and such that each $K_i = \mathrm{Coker}(E_{i+1} \to E_i)$ is G-injective. Let $\mathfrak{a}$ be an ideal of R. If we apply the functor $\Gamma_{\mathfrak{a}}(-)$ to the sequence above and we use [11, Theorem 3.1], we get the following exact sequence

$$\cdots \to \Gamma_{\mathfrak{a}}(E_2) \to \Gamma_{\mathfrak{a}}(E_1) \to \Gamma_{\mathfrak{a}}(E_0) \to \Gamma_{\mathfrak{a}}(M) \to 0$$

where by basic properties of the section functor, each $\Gamma_{\mathfrak{a}}(E_i)$ is injective. Let F be an R-module of finite flat dimension. It follows from [10, p. 84] that $\mathrm{pd}_R F \leq d$. Thus there are the following isomorphisms $\mathrm{Ext}^1_R(F, \Gamma_{\mathfrak{a}}(M)) \cong \mathrm{Ext}^2_R(F, \Gamma_{\mathfrak{a}}(K_1)) \cong \cdots \cong \mathrm{Ext}^{d+1}_R(F, \Gamma_{\mathfrak{a}}(K_d)) = 0$ and this proves the assertion. $\square$

The above proposition gives us a vanishing result for generalized local cohomology. This is stated in the following corollary:

Corollary 3.6. *Let R be any ring of Krull dimension d, and let M be a finitely generated R-module of finite projective dimension. If N is an R-module of finite G-injective dimension with $\mathrm{Gid}_R N = t$, then $H^i_{\mathfrak{a}}(M, N) = 0$ for any $i > t$ and any ideal $\mathfrak{a}$ of R. In particular, if N is G-injective, then $H^i_{\mathfrak{a}}(M, N) = 0$ for any $i > 0$ and any ideal $\mathfrak{a}$ of R.*

Proof. According to Proposition 3.2, there is an isomorphism

$$H^{t+j}_{\mathfrak{a}}(M, N) \cong \mathrm{Ext}^j_R(M, \Gamma_{\mathfrak{a}}(\Omega^t(N)))$$

where $\Omega^t(N)$ is t-th cosyzygy of an injective resolution of N. By using [6, Lemma 4.2], $\Omega^t(N)$ is G-injective, and hence it follows from Proposition 3.5 that $\Gamma_{\mathfrak{a}}(\Omega^t(N))$ is strongly cotorsion for any ideal $\mathfrak{a}$ of R. Now the result follows by the definition of strong cotorsion. The second part is trivial when $t = 0$. $\square$

Theorem 3.7. *Let R be any ring of Krull dimension d, let $\mathfrak{a}$ be any ideal of R, and let M be finitely generated R-module of finite projective dimension. If N is any R-module, then for each i, $H^i_{\mathfrak{a}}(M, N)$ is computed by applying the section functor $\Gamma_{\mathfrak{a}}(M, -)$ to any G-injective resolution of N.*

Proof. Let $0 \to N \to G^0 \to G^1 \to G^2 \to G^3 \to \cdots$ be an arbitrary G-injective resolution of N. We prove the assertion by induction on i. If $i = 0$, the result is trivial. Consider $i = 1$ and $N^j = \mathrm{Ker}(G^j \to G^{j+1})$ for each j. By using Corollary 3.6, there exists the following exact sequence of R-modules

$$0 \to \Gamma_{\mathfrak{a}}(M, N) \to \Gamma_{\mathfrak{a}}(M, G^0) \to \Gamma_{\mathfrak{a}}(M, N^1) \to H^1_{\mathfrak{a}}(M, N) \to 0.$$

We note that $H^1_{\mathfrak{a}}(M, N) \cong \Gamma_{\mathfrak{a}}(M, N^1)/\mathrm{Im}(\Gamma_{\mathfrak{a}}(M, G^0) \to \Gamma_{\mathfrak{a}}(M, N^1)) = \mathrm{Ker}(\Gamma_{\mathfrak{a}}(M, G^1) \to \Gamma_{\mathfrak{a}}(M, G^2))/\mathrm{Im}(\Gamma_{\mathfrak{a}}(M, G^0) \to \Gamma_{\mathfrak{a}}(M, G^1))$; and hence the result follows in this case. Now, suppose inductively that the result has been proved for all values smaller than i, and we prove it for i. As we have an isomorphism $H^i_{\mathfrak{a}}(M, N) \cong H^{i-1}_{\mathfrak{a}}(M, N^1)$, using the induction hypotheses for $i-1$, we have $H^{i-1}_{\mathfrak{a}}(M, N^1) \cong \mathrm{Ker}(\Gamma_{\mathfrak{a}}(M, G^i) \to \Gamma_{\mathfrak{a}}(M, G^{i+1}))/\mathrm{Im}(\Gamma_{\mathfrak{a}}(M, G^{i-1}) \to \Gamma_{\mathfrak{a}}(M, G^i))$; and this gives the assertion. □

In the following theorem, we will get a new characterization for a Gorenstein local ring.

Theorem 3.8. *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension d. The following conditions are equivalent.*

(i) *any $\mathfrak{m}$-torsion strongly cotorsion R-module is strongly copure injective.*
(ii) $\mathrm{cid}_R N \le d$ *for any $\mathfrak{m}$-torsion R-module N.*
(iii) $\mathrm{cfd}_R M \le d$ *for any finitely generated R-module M.*
(iv) *R is Gorenstein.*

Proof. (i) ⇒ (ii). Let N be an $\mathfrak{m}$-torsion R-module. So, there exists an injective resolution $0 \to N \to E^0 \to E^1 \to \cdots$ such that each E^i is $\mathfrak{m}$-torsion. Let $\Omega^i(N) = \mathrm{Ker}(E^i \to E^{i+1})$ be i-th cosyzygy of this resolution, and let F be an R-module of finite flat dimension. It follows from [10, p. 84] that $\mathrm{pd}_R F \le d$; hence we have $\mathrm{Ext}^1_R(F, \Omega^d(N)) \cong \mathrm{Ext}^{d+1}_R(F, N) = 0$. Thus $\Omega^d(N)$ is $\mathfrak{m}$-torsion and strongly cotorsion, and so $\Omega^d(N)$ is strongly copure injective. Therefore $\mathrm{cid}_R N \le d$. (ii)⇒(iii). Let M be a finitely generated R-module and $D(-) = \mathrm{Hom}_R(-, E(R/\mathfrak{m}))$ be Matlis duality functor. Then $D(M)$ is Artinian, and hence it is $\mathfrak{m}$-torsion. In view of (ii), there is $\mathrm{cid}_R D(M) \le d$. Now, using [4, Lemma 3.4], we have $\mathrm{cfd}_R M = \mathrm{cid}_R D(M) \le d$. (iii)⇒(iv) follows from [4, Theorem 4.1]. (iv)⇒ (i). Since R is Gorenstein, one can easily show that any strongly cotorsion R-module S is G-injective and so is strongly copure injective. □

In the following theorem, we will get a new characterization for regular local rings.

Theorem 3.9. *Let $(R, \mathfrak{m}, k)$ be a Gorenstein local ring with residue field k. The following conditions are equivalent:*

(i) $H^i_{\mathfrak{m}}(M, G) = 0$ *for all $i > 0$ and all finitely generated R-modules M and all G-injective R-modules G.*
(ii) $H^i_{\mathfrak{m}}(R/\mathfrak{a}, G) = 0$ *for all $i > 0$ and all ideals $\mathfrak{a}$ of R and all G-injective R-modules G.*
(iii) *Any $\mathfrak{m}$-torsion R-module has finite injective dimension.*
(iv) $\mathrm{id}_R k < \infty$.
(v) *R is regular.*

Proof. (i) ⇒ (ii) is trivial. (ii) ⇒ (iii). Let G be a G-injective R-module. In view of Proposition 3.1 and the hypotheses, we have $H^i_{\mathfrak{m}}(R/\mathfrak{a}, G) \cong \mathrm{Ext}^i(R/\mathfrak{a}, \Gamma_{\mathfrak{m}}(G)) = 0$ for all $i > 0$. This means that $\Gamma_{\mathfrak{m}}(G)$ is injective. Thus, for any G-injective R-module G, the module $\Gamma_{\mathfrak{m}}(G)$ is injective. Let N be an $\mathfrak{m}$-torsion. As R is Gorenstein, N has finite G-injective dimension. Let $\mathrm{Gid}_R N = t$. Since N is $\mathfrak{m}$-torsion, there exists an injective resolution $0 \to N \to E^0 \to E^1 \to \cdots$ of N such that each E^i is $\mathfrak{m}$-torsion. We note that $\Omega^t(N)$ is G-injective and $\mathfrak{m}$-torsion by [6, Lemma 2.4]. Now, the previous argument implies that $\Gamma_{\mathfrak{m}}(\Omega^t(N)) = \Omega^t(N)$ is injective; hence $\mathrm{id}_R N \le t$. (iii) ⇒ (iv) is trivial by the fact that k is $\mathfrak{m}$-torsion. (iv) ⇒ (v). Since $\mathrm{id}_R k < \infty$, by using [5, Theorem 9.1.10], we have $\mathrm{pd}_R k < \infty$. Now the result follows by [13, Theorem 4.4.16]. (v) ⇒ (i). As R is regular, any finitely generated R-module has finite projective dimension. Now, the assertion follows by Corollary 3.6. □

4. Foxby classes and section functor

We begin this section with the following lemma.

Lemma 4.1. *Let $\mathfrak{a}$ be an ideal of R such that $\Gamma_{\mathfrak{a}}(G)$ is copure injective for any G-injective R-module G. Then $\Gamma_{\mathfrak{a}}(G)$ is G-injective for any G-injective module G.*

Proof. Assume that G is a G-injective R-module. So there exists a $\mathrm{Hom}(\mathcal{I}, -)$ exact exact sequence

$$\cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$$

of injective R-modules such that $G = \operatorname{Ker}(E^0 \to E^1)$. Applying the functor $\Gamma_{\mathfrak{a}}(-)$ to this exact sequence and using [11, Theorem 3.1], we will get the following exact sequence of injective R-modules

$$\cdots \to \Gamma_{\mathfrak{a}}(E_1) \to \Gamma_{\mathfrak{a}}(E_0) \to \Gamma_{\mathfrak{a}}(E^0) \to \Gamma_{\mathfrak{a}}(E^1) \to \cdots$$

where $\Gamma_{\mathfrak{a}}(G) = \operatorname{Ker}(\Gamma_{\mathfrak{a}}(E^0) \to \Gamma_{\mathfrak{a}}(E^1))$. By using the hypothesis, for any injective R-module E, the functor $\operatorname{Hom}_R(E, -)$ leaves the above exact sequence exact. Thus $\Gamma_{\mathfrak{a}}(G)$ is G-injective. $\square$

A proof similar to that in Proposition 3.5 shows that over a local ring $(R, \mathfrak{m})$ with Krull dimension d, the G-injective R-modules are strongly cotorsion. So now, by considering this fact, we have the following proposition.

Proposition 4.2. *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring, and let $\Gamma_{\mathfrak{m}}(G)$ be copure injective for all G-injective R-modules G. If N is a strongly cotorsion R-module such that $N \in \widetilde{\mathcal{GI}}$, then $\Gamma_{\mathfrak{m}}(N) \in \widetilde{\mathcal{GI}}$.*

Proof. We proceed by induction on $\operatorname{Gid}_R(N) = n$. If $n = 0$, then N is G-injective; hence the result follows by Lemma 4.1. Now, suppose inductively that the result has been proved for all values smaller than n, and so we prove it for n. As $\operatorname{Gid}_R(N) = n$, there is an exact sequence $0 \to N \to G \to K \to 0$ of R-modules such that G is G-injective and $\operatorname{Gid}_R(K) \le n - 1$. By applying the functor $\Gamma_{\mathfrak{m}}(-)$ to the above exact sequence and using [12, Theorem 2.2], we get the following exact sequence of R-modules

$$0 \to \Gamma_{\mathfrak{m}}(N) \to \Gamma_{\mathfrak{m}}(G) \to \Gamma_{\mathfrak{m}}(K) \to 0.$$

One can easily see that that K is strongly cotorsion, and then we can use the induction hypothesis for K and conclude that $\Gamma_{\mathfrak{m}}(K) \in \widetilde{\mathcal{GI}}$. Now, since by Lemma 4.1, the module $\Gamma_{\mathfrak{m}}(G)$ is G-injective, the preceding exact sequence implies that $\Gamma_{\mathfrak{m}}(N) \in \widetilde{\mathcal{GI}}$. $\square$

Definition 4.3. Let R be a Cohen–Macaulay local ring of Krull dimension d admitting a dualizing module ω and with residue field k. Following [7], $\mathcal{G}_0(R)$ denote the class of R-modules M such that $\operatorname{Tor}_i^R(\omega, M) = \operatorname{Ext}^i_R(\omega, \omega \otimes_R M) = 0$ for all $i > 0$, and such that the natural map $M \to \operatorname{Hom}(\omega, \omega \otimes_R M)$ is an isomorphism. This class of R-modules is called the *Auslander class*. Also, $\mathcal{J}_0(R)$ denote the class of R-modules N such that $\operatorname{Ext}^i_R(\omega, N) = \operatorname{Tor}_i^R(\omega, \operatorname{Hom}_R(\omega, N)) = 0$ for all $i > 0$, and such that the natural map $\omega \otimes_R \operatorname{Hom}_R(\omega, N) \to N$ is an isomorphism. This class of R-modules is called the *Bass class*. It should be noted that $\mathcal{G}_0(R)$ and $\mathcal{J}_0(R)$ are also called *Foxby classes*.

It follows from [5, Proposition 10.4.23] that $\mathcal{J}_0(R) = \widetilde{\mathcal{GI}}$. Moreover, according to [5, Proposition 10.4.17 and Corollary 10.4.29] we have $\mathcal{G}_0(R) \subseteq \widetilde{\mathcal{GF}}$, and according to [5, Theorems 10.4.10 and 10.4.28] we have $\widetilde{\mathcal{GF}} \subseteq \mathcal{G}_0(R)$. Therefore, we can deduce that $\mathcal{G}_0(R) = \widetilde{\mathcal{GF}}$. When R is Gorenstein, each of the classes $\mathcal{G}_0(R)$ and $\mathcal{J}_0(R)$ contains all R-modules.

In the rest of this section, we assume that $(R, \mathfrak{m})$ is a Cohen–Macaulay local ring of Krull dimension d admitting a dualizing module ω.

Lemma 4.4. *Let $x_1, \ldots, x_t \in R$ be an R-sequence or an ω-sequence. Then*

$$\operatorname{id}_R(\omega/(x_1, \ldots, x_t)\omega) < \infty.$$

Proof. At first, assume that $x_1, \ldots, x_t$ is an R-sequence. Let $t = 1$. As x_1 is an R-sequence, there is an exact sequence of R-modules $0 \to R \xrightarrow{x_1\cdot} R \to R/x_1R \to 0$. On the other hand, since ω is maximal Cohen–Macaulay, by [3, Theorem 2.8], it is strongly torsion free. Thus, applying the functor $\omega \otimes_R -$ to the exact sequence above gives the following exact sequence $0 \to \omega \xrightarrow{x_1\cdot} \omega \to \omega/x_1\omega \to 0$. So one can easily see that $\operatorname{id}_R(\omega/x_1\omega) < \infty$. Finally, an easy induction on t completes the proof. Now, assume that $x_1, \ldots, x_t$ is an ω-sequence. If $t = 1$, there exists an exact sequence of R-modules $0 \to \omega \xrightarrow{x_1\cdot} \omega \to \omega/x_1\omega \to 0$; and hence $\operatorname{id}_R(\omega/x_1\omega) < \infty$. If $t > 1$, the assertion follows using an induction argument similar to that in the first part. $\square$

Lemma 4.5. *Let $\mathfrak{a}$ be an ideal of R generated by an R-sequence or an ω-sequence. Then $H^i_{\mathfrak{a}}(\omega, G) = 0$ for all $i > 0$ and all G-injective R-modules G.*

Proof. Let G be a G-injective R-module, and let $\mathfrak{a} = (x_1, \dots, x_t)R$ be such that $x_1, \dots, x_t \in R$ is an R-sequence or an ω-sequence. In view of the previous lemma and using [5, Proposition 10.1.3], we have

$$H^i_{\mathfrak{a}}(\omega, G) \cong \varinjlim_{(\alpha_1,\dots,\alpha_t)} \mathrm{Ext}^i_R(\omega/(x_1^{\alpha_1}, \dots, x_t^{\alpha_t})\omega, G) = 0. \quad \square$$

Proposition 4.6. *Let $\mathfrak{a}$ be an ideal of R such that $H^1_{\mathfrak{a}}(\omega, G) = 0$ for all G-injective R-modules G. Then $\Gamma_{\mathfrak{a}}(G) \in \mathcal{J}_0(R)$ for all G-injective modules G.*

Proof. Let G be any G-injective R-module. So there is an exact sequence of R-modules $0 \to G \to E \to K \to 0$ such that E is injective and K is Gorenstein injective. If we apply the functor $H^i_{\mathfrak{a}}(\omega, -)$ to this exact sequence and use the hypothesis, we can deduce that $H^2_{\mathfrak{a}}(\omega, G) = 0$. Replacing G by K and using a similar argument for any $i \geq 2$, we finally deduce that $H^i_{\mathfrak{a}}(\omega, G) = 0$ for all $i > 0$. Now, according to Proposition 3.1, for each $i > 0$, we have $H^i_{\mathfrak{a}}(\omega, G) \cong \mathrm{Ext}^i_R(\omega, \Gamma_{\mathfrak{a}}(G)) = 0$. On the other hand, since G is G-injective, there exists a $\mathrm{Hom}(\mathcal{I}, -)$ exact exact sequence

$$\cdots \to E_1 \to E_0 \to E^0 \to E^1 \cdots$$

of injective R-modules such that $G = \mathrm{Ker}(E^0 \to E^1)$, and by the same reasoning mentioned in the proof of Lemma 4.1, we have the following exact sequence of injective R-modules

$$\cdots \to \Gamma_{\mathfrak{a}}(E_1) \to \Gamma_{\mathfrak{a}}(E_0) \to \Gamma_{\mathfrak{a}}(E^0) \to \Gamma_{\mathfrak{a}}(E^1) \cdots$$

such that $\Gamma_{\mathfrak{a}}(G) = \mathrm{Ker}(\Gamma_{\mathfrak{a}}(E^0) \to \Gamma_{\mathfrak{a}}(E^1))$. It should be noted that the modules $K_0 = \mathrm{Ker}(E_0 \to E^0)$ and $K_i = \mathrm{Ker}(E_i \to E_{i-1})$ are G-injective for all $i > 0$. In view of the first argument, the following diagram commutes:

$$\begin{tikzcd}
\omega \otimes_R \mathrm{Hom}_R(\omega, \Gamma_{\mathfrak{a}}(E_1)) \arrow[r] \arrow[d] & \omega \otimes_R \mathrm{Hom}_R(\omega, \Gamma_{\mathfrak{a}}(E_0)) \arrow[r] \arrow[d] & \omega \otimes_R \mathrm{Hom}_R(\omega, \Gamma_{\mathfrak{a}}(G)) \arrow[r] \arrow[d] & 0 \\
\Gamma_{\mathfrak{a}}(E_1) \arrow[r] & \Gamma_{\mathfrak{a}}(E_0) \arrow[r] & \Gamma_{\mathfrak{a}}(G) \arrow[r] & 0
\end{tikzcd}$$

Since $\Gamma_{\mathfrak{a}}(E_i) \in \mathcal{J}_0(R)$, the two vertical maps on the left are isomorphisms, and so we have an isomorphism $\Gamma_{\mathfrak{a}}(G) \cong \omega \otimes_R \mathrm{Hom}_R(\omega, \Gamma_{\mathfrak{a}}(G))$. Now, it remains to show that, for each $i > 0$, we have $\mathrm{Tor}^R_i(\omega, \mathrm{Hom}_R(\omega, \Gamma_{\mathfrak{a}}(G))) = 0$. Consider the exact sequence $0 \to \Gamma_{\mathfrak{a}}(K_0) \to \Gamma_{\mathfrak{a}}(E_0) \to \Gamma_{\mathfrak{a}}(G) \to 0$. Since K_0 is G-injective, by the previous arguments there is an isomorphism $\Gamma_{\mathfrak{a}}(K_0) \cong \omega \otimes_R \mathrm{Hom}_R(\omega, \Gamma_{\mathfrak{a}}(K_0))$. On the other hand, using the first argument mentioned in this proof and applying the functor $\mathrm{Hom}_R(\omega, -)$ to the above exact sequence, we get the exact sequence $0 \to \mathrm{Hom}_R(\omega, \Gamma_{\mathfrak{a}}(K_0)) \to \mathrm{Hom}_R(\omega, \Gamma_{\mathfrak{a}}(E_0)) \to \mathrm{Hom}_R(\omega, \Gamma_{\mathfrak{a}}(G)) \to 0$. Now, by applying the functor $\omega \otimes_R -$ to the above exact sequence and using this fact that $\Gamma_{\mathfrak{a}}(E_0) \in \mathcal{J}_0(R)$, one can easily see that $\mathrm{Tor}^R_1(\omega, \mathrm{Hom}_R(\omega, \Gamma_{\mathfrak{a}}(G))) = 0$. We get the same result for $\Gamma_{\mathfrak{a}}(K_0)$, and so we can deduce that $\mathrm{Tor}^R_2(\omega, \mathrm{Hom}_R(\omega, \Gamma_{\mathfrak{a}}(G))) = 0$. Now, an easy induction on i completes the proof. $\square$

Theorem 4.7. *Let $\mathfrak{a}$ be an ideal of R such that $H^1_{\mathfrak{a}}(\omega, G) = 0$ for any G-injective module G. Then $\Gamma_{\mathfrak{a}}(G)$ is G-injective for any G-injective module G.*

Proof. Let G be a G-injective R-module. Then there exists an exact sequence of R-modules

$$\cdots \to E_2 \to E_1 \to E_0 \to G \to 0$$

such that each E_i is injective and such that each $K_i = \mathrm{Ker}(E_i \to E_{i-1})$ is G-injective. Applying the functor $\Gamma_{\mathfrak{a}}(-)$ to the exact sequence above and using [11, Theorem 3.1], we get the following exact sequence:

$$0 \to \Gamma_{\mathfrak{a}}(K_{d-1}) \to \Gamma_{\mathfrak{a}}(E_{d-1}) \to \cdots \to \Gamma_{\mathfrak{a}}(E_1) \to \Gamma_{\mathfrak{a}}(E_0) \to \Gamma_{\mathfrak{a}}(G) \to 0.$$

According to Proposition 4.6, we have $\Gamma_{\mathfrak{a}}(K_{d-1}) \in \mathcal{J}_0(R)$ and $\Gamma_{\mathfrak{a}}(G)$ is d-th cosyzygy of $\Gamma_{\mathfrak{a}}(K_{d-1})$. Now, since $\mathrm{id}_R(\omega) \leq d$, by using [5, Theorem 12.4.1], the module $\Gamma_{\mathfrak{a}}(G)$ is G-injective. $\square$

Corollary 4.8. *Let $\mathfrak{a}$ be an ideal of R generated by an R-sequence or an ω-sequence. Then $\Gamma_{\mathfrak{a}}(G)$ is G-injective for any G-injective module G. In particular, $\Gamma_{\mathfrak{m}}(G)$ is G-injective for any G-injective module G.*

Proof. It follows from Lemma 4.5 that $H^i_{\mathfrak{a}}(\omega, G) = 0$ for all $i > 0$ and all G-injective modules G. Then the claim follows by the previous theorem. For the second assertion, let $x_1, \ldots, x_d \in \mathfrak{m}$ be a system of parameters of R. Since R is Cohen–Macaulay, this is an R-sequence. On the other hand, since for each R-module N and each i there are the isomorphisms $H^i_{\mathfrak{m}}(\omega, N) \cong H^i_{(x_1,\ldots,x_d)R}(\omega, N)$ and $H^i_{\mathfrak{m}}(N) \cong H^i_{(x_1,\ldots,x_d)R}(N)$, we may assume that $\mathfrak{m} = (x_1, \ldots, x_d)R$. Now, the assertion follows by the first part. □

Proposition 4.9. *If N is a strongly cotorsion R-module such that $N \in \mathcal{J}_0(R)$, then $\Gamma_{\mathfrak{m}}(N) \in \mathcal{J}_0(R)$.*

Proof. The result follows by the previous corollary and a proof similar to that of Proposition 4.2. □

Lemma 4.10. *Let M be a G-flat R-module. Then $H^i_{\mathfrak{m}}(M) = 0$ for all $i < d$.*

Proof. As M is G-flat, there exists an exact sequence of R-modules $0 \to M \to F \to K \to 0$ such that F is flat and K is G-flat. Application of the functor $\Gamma_{\mathfrak{m}}(-)$ to this exact sequence induces the following exact sequence:

$$0 \to \Gamma_{\mathfrak{m}}(M) \to \Gamma_{\mathfrak{m}}(F) \to \Gamma_{\mathfrak{m}}(K) \to H^1_{\mathfrak{m}}(M) \to \cdots.$$

Since F is flat and R is Cohen–Macaulay, for all $i < d$, we have $H^i_{\mathfrak{m}}(F) \cong H^i_{\mathfrak{m}}(R) \otimes_R F = 0$. This fact implies that $\Gamma_{\mathfrak{m}}(M) = 0$, and for any $1 \leq i < d$, there is an isomorphism $H^i_{\mathfrak{m}}(M) \cong H^{i-1}_{\mathfrak{m}}(K)$. If we replace M by K, we deduce that $H^1_{\mathfrak{m}}(M) = 0$. Now, by using an easy induction in i, we finish the proof. □

Lemma 4.11. *Let M be a G-flat R-module. Then $H^i_{\mathfrak{m}}(\omega \otimes_R M) = 0$ for all $i < d$.*

Proof. Since M is G-flat, there exists an exact sequence of R-modules $0 \to M \to F \to K \to 0$ such that F is flat, and such that K is G-flat too. Applying the functor $\omega \otimes_R -$ to this exact sequence and using [5, Theorem 10.4.31] induces the following exact sequence of R-modules:

$$0 \to \omega \otimes_R M \to \omega \otimes_R F \to \omega \otimes_R K \to 0.$$

If we apply the functor $\Gamma_{\mathfrak{m}}(-)$ to the above exact sequence, we get the following long exact sequence:

$$0 \to \Gamma_{\mathfrak{m}}(\omega \otimes_R M) \to \Gamma_{\mathfrak{m}}(\omega \otimes_R F) \to \Gamma_{\mathfrak{m}}(\omega \otimes_R K) \to H^1_{\mathfrak{m}}(\omega \otimes_R M) \to H^1_{\mathfrak{m}}(\omega \otimes_R F_0) \to \cdots.$$

As ω is maximal Cohen–Macaulay, for each $i < d$, we have $H^i_{\mathfrak{m}}(\omega \otimes_R F) \cong H^i_{\mathfrak{m}}(\omega) \otimes_R F = 0$. This fact implies that $\Gamma_{\mathfrak{m}}(\omega \otimes_R M) = 0$, and for each $0 < i < d$, there is an isomorphism $H^i_{\mathfrak{m}}(\omega \otimes_R M) \cong H^{i-1}_{\mathfrak{m}}(\omega \otimes_R K)$. If we replace M by K, we deduce that $H^1_{\mathfrak{m}}(\omega \otimes_R M) = 0$. Now, using an easy induction on i, the result follows. □

Theorem 4.12. *Let M be a G-flat R-module. Then $H^d_{\mathfrak{m}}(M) \in \mathcal{G}_0(R)$.*

Proof. We prove that the natural morphism $H^d_{\mathfrak{m}}(M) \to \mathrm{Hom}_R(\omega, \omega \otimes_R H^d_{\mathfrak{m}}(M))$ is an isomorphism. We first prove the assertion for the case $M = R$. It follows from [1, Theorem 3.3.5] that $\omega \otimes_R \widehat{R}$ is dualizing module of $\widehat{R}$. On the other hand, by a basic theorem of local cohomology, the modules $H^d_{\mathfrak{m}}(R)$ and $H^d_{\mathfrak{m}}(\omega)$ are Artinian. Also, by the local duality theorem of local cohomology, there are the isomorphisms $H^d_{\mathfrak{m}\widehat{R}}(\widehat{R}) \cong \mathrm{Hom}_{\widehat{R}}(\omega \otimes_R \widehat{R}, E(R/\mathfrak{m}))$ and $H^d_{\mathfrak{m}\widehat{R}}(\omega \otimes_R \widehat{R}) \cong E(R/\mathfrak{m})$. So we have the following isomorphisms:

$$\begin{aligned} H^d_{\mathfrak{m}}(R) &\cong H^d_{\mathfrak{m}}(R) \otimes_R \widehat{R} \cong H^d_{\mathfrak{m}\widehat{R}}(\widehat{R}) \cong \mathrm{Hom}_{\widehat{R}}(\omega \otimes_R \widehat{R}, E(R/\mathfrak{m})) \\ &\cong \mathrm{Hom}_{\widehat{R}}(\omega \otimes_R \widehat{R}, H^d_{\mathfrak{m}\widehat{R}}(\omega \otimes_R \widehat{R})) \cong \mathrm{Hom}_R(\omega, H^d_{\mathfrak{m}}(\omega)) \otimes_R \widehat{R} \\ &\cong \mathrm{Hom}_R(\omega, H^d_{\mathfrak{m}}(\omega)) \cong \mathrm{Hom}_R(\omega, \omega \otimes_R H^d_{\mathfrak{m}}(R)). \end{aligned}$$

The two last isomorphisms hold because $\mathrm{Hom}_R(\omega, H^d_{\mathfrak{m}}(\omega))$ is Artinian and $H^d_{\mathfrak{m}}(-)$ is a right exact functor (see [2, Exercise 6.1.9]). Now, for any flat R-module F, there are the isomorphisms $\mathrm{Hom}_R(\omega, \omega \otimes_R H^d_{\mathfrak{m}}(F)) \cong \mathrm{Hom}_R(\omega, \omega \otimes_R H^d_{\mathfrak{m}}(R) \otimes_R F) \cong \mathrm{Hom}_R(\omega, \omega \otimes_R H^d_{\mathfrak{m}}(R)) \otimes_R F \cong H^d_{\mathfrak{m}}(R) \otimes_R F \cong H^d_{\mathfrak{m}}(F)$. Since M is G-flat, there is an exact sequence of R-modules $0 \to M \to F_0 \to F_1$ of R-modules such that F_0 and F_1 are flat and $K = \mathrm{Im}(F_0 \to F_1)$ is G-flat. Applying the functor $H^d_{\mathfrak{m}}(-)$ to the above exact sequence, and using Lemma 4.10, gives the following exact sequence $0 \to H^d_{\mathfrak{m}}(M) \to H^d_{\mathfrak{m}}(F_0) \to H^d_{\mathfrak{m}}(F_1)$. Now, in view of Lemma 4.11 and recalling that for any R-module N there is an isomorphism $\omega \otimes_R H^d_{\mathfrak{m}}(N) \cong H^d_{\mathfrak{m}}(\omega \otimes_R N)$, we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & H^d_{\mathfrak{m}}(M) \arrow[r] \arrow[d] & H^d_{\mathfrak{m}}(F_0) \arrow[r] \arrow[d] & H^d_{\mathfrak{m}}(F_1) \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}_R(\omega, \omega \otimes_R H^d_{\mathfrak{m}}(M)) \arrow[r] & \mathrm{Hom}_R(\omega, \omega \otimes_R H^d_{\mathfrak{m}}(F_0)) \arrow[r] & \mathrm{Hom}_R(\omega, \omega \otimes_R H^d_{\mathfrak{m}}(F_1))
\end{tikzcd}$$

By the argument above, the two vertical maps on the right are isomorphisms, and so we will have an isomorphism $H^d_{\mathfrak{m}}(M) \cong \mathrm{Hom}_R(\omega, \omega \otimes_R H^d_{\mathfrak{m}}(M))$. Now, we prove that $\mathrm{Tor}^R_i(\omega, H^d_{\mathfrak{m}}(M)) = 0$ for all $i > 0$. First, we prove the assertion when $M = F$ is flat. In view of [13, Proposition 3.2.9] and of the local duality theorem for local cohomology, for any $i > 0$, there are the isomorphisms $\mathrm{Tor}^R_i(\omega, H^d_{\mathfrak{m}}(R)) \cong \mathrm{Tor}^R_i(\omega, H^d_{\mathfrak{m}}(R) \otimes_R \widehat{R}) \cong \mathrm{Tor}^{\widehat{R}}_i(\omega \otimes_R \widehat{R}, H^d_{\mathfrak{m}\widehat{R}}(\widehat{R})) \cong \mathrm{Tor}^{\widehat{R}}_i(\omega \otimes_R \widehat{R}, \mathrm{Hom}_{\widehat{R}}(\omega \otimes_R \widehat{R}, E(R/\mathfrak{m}))) \cong \mathrm{Hom}_{\widehat{R}}(\mathrm{Ext}^i_{\widehat{R}}(\omega \otimes_R \widehat{R}, \omega \otimes_R \widehat{R}), E(R/\mathfrak{m})) = 0$. Let P be a finitely generated projective R-module. Then, for some n, we have $P \cong \oplus_n R$, and hence by the argument above, for each $j > 0$, we have $\mathrm{Tor}^R_i(\omega, H^d_{\mathfrak{m}}(P)) \cong \oplus_n \mathrm{Tor}^R_i(\omega, H^d_{\mathfrak{m}}(R)) = 0$. Now, if F is any flat R-module, then we have $F \cong \varinjlim_i P_i$, where P_i is a finitely generated projective R-module for each i. Thus, by [8, p. 279], for each $j > 0$, we have $\mathrm{Tor}^R_j(\omega, H^d_{\mathfrak{m}}(F)) \cong \mathrm{Tor}^R_j(\omega, H^d_{\mathfrak{m}}(\varinjlim_i P_i)) \cong \varinjlim_i \mathrm{Tor}^R_j(\omega, H^d_{\mathfrak{m}}(P_i)) = 0$. Now, we prove the claim for the G-flat module M. As M is G-flat, there is an exact sequence of R-modules

$$0 \to K^0 \to F^0 \to M \to 0 \tag{$\star$}$$

such that F^0 is flat and K^0 is G-flat. Now, if we apply the functor $\omega \otimes_R -$ to the above exact sequence, and then we apply the functor $H^d_{\mathfrak{m}}(-)$ and use Lemma 4.11, we will get the following exact sequence:

$$0 \to H^d_{\mathfrak{m}}(\omega \otimes_R K^0) \to H^d_{\mathfrak{m}}(\omega \otimes_R F^0) \to H^d_{\mathfrak{m}}(\omega \otimes_R M) \to 0.$$

So we will have the following exact sequence

$$0 \to \omega \otimes_R H^d_{\mathfrak{m}}(K^0) \to \omega \otimes_R H^d_{\mathfrak{m}}(F^0) \to \omega \otimes_R H^d_{\mathfrak{m}}(M) \to 0. \tag{$\star\star$}$$

On the other hand, applying the functor $H^d_{\mathfrak{m}}(-)$ to the exact sequence ($\star$) and using Lemma 4.10, we have the exact sequence

$$0 \to H^d_{\mathfrak{m}}(K^0) \to H^d_{\mathfrak{m}}(F^0) \to H^d_{\mathfrak{m}}(M) \to 0.$$

Now applying the functor $\omega \otimes_R -$ to the above exact sequence and in view of the exact sequence ($\star\star$), we will have the following long exact sequence:

$$\cdots \to \mathrm{Tor}^R_1(\omega, H^d_{\mathfrak{m}}(K^0)) \to \mathrm{Tor}^R_1(\omega, H^d_{\mathfrak{m}}(F^0)) \to \mathrm{Tor}^R_1(\omega, H^d_{\mathfrak{m}}(M)) \to 0$$
$$\mathrm{Tor}^R_i(\omega, H^d_{\mathfrak{m}}(F^0)) \to \mathrm{Tor}^R_i(\omega, H^d_{\mathfrak{m}}(M)) \to \mathrm{Tor}^R_{i-1}(\omega, H^d_{\mathfrak{m}}(K^0)) \to \mathrm{Tor}^R_{i-1}(\omega, H^d_{\mathfrak{m}}(F^0)).$$

This fact that $\mathrm{Tor}^R_i(\omega, H^d_{\mathfrak{m}}(F)) = 0$ for any flat module F and any $i > 0$ implies that $\mathrm{Tor}^R_1(\omega, H^d_{\mathfrak{m}}(M)) = 0$, and that for each $i > 1$ there is an isomorphism $\mathrm{Tor}^R_i(\omega, H^d_{\mathfrak{m}}(M)) \cong \mathrm{Tor}^R_{i-1}(\omega, H^d_{\mathfrak{m}}(K^0))$. Replacing M by K^0, we conclude that $\mathrm{Tor}^R_2(\omega, H^d_{\mathfrak{m}}(M)) = 0$. Now, using an easy induction on i, we can deduce that $\mathrm{Tor}^R_i(\omega, H^d_{\mathfrak{m}}(M)) = 0$ for all $i > 0$. It remains to show that $\mathrm{Ext}^i_R(\omega, \omega \otimes_R H^d_{\mathfrak{m}}(M)) = 0$ for all $i > 0$. If we apply the functor $H^d_{\mathfrak{m}}(-)$ to the exact sequence $0 \to M \to F_0 \to K \to 0$ mentioned above, and using Lemma 4.10, we get the exact sequence

$$0 \to H^d_{\mathfrak{m}}(M) \to H^d_{\mathfrak{m}}(F_0) \to H^d_{\mathfrak{m}}(K) \to 0. \tag{$\bullet$}$$

Now, applying the functor $\omega \otimes_R$, and using this fact that $\mathrm{Tor}^R_1(\omega, H^d_{\mathfrak{m}}(K)) = 0$, we will have the following exact sequence

$$0 \to \omega \otimes_R H^d_{\mathfrak{m}}(M) \to \omega \otimes_R H^d_{\mathfrak{m}}(F_0) \to \omega \otimes_R H^d_{\mathfrak{m}}(K) \to 0.$$

If we apply the functor $\mathrm{Hom}(\omega, -)$ to this exact sequence, we get the following exact sequence

$$\begin{aligned}0 &\to \operatorname{Hom}(\omega, \omega \otimes_R H^d_{\mathfrak{m}}(M)) \to \operatorname{Hom}(\omega, \omega \otimes_R H^d_{\mathfrak{m}}(F_0)) \to \operatorname{Hom}(\omega, \omega \otimes_R H^d_{\mathfrak{m}}(K))\\ &\to \operatorname{Ext}^1_R(\omega, \omega \otimes_R H^d_{\mathfrak{m}}(M)) \to \cdots \to \operatorname{Ext}^{i-1}_R(\omega, \omega \otimes_R H^d_{\mathfrak{m}}(F_0)) \to \operatorname{Ext}^{i-1}_R(\omega, \omega \otimes_R H^d_{\mathfrak{m}}(K))\\ &\to \operatorname{Ext}^i_R(\omega, \omega \otimes_R H^d_{\mathfrak{m}}(M)) \to \operatorname{Ext}^i_R(\omega, \omega \otimes_R H^d_{\mathfrak{m}}(F_0)) \to \cdots. \end{aligned} \tag{••}$$

As $H^d_{\mathfrak{m}}(M) \cong \operatorname{Hom}(\omega, \omega \otimes_R H^d_{\mathfrak{m}}(M))$ for any G-flat module M and $\operatorname{Ext}^i_R(\omega, \omega \otimes_R H^d_{\mathfrak{m}}(F)) \cong \operatorname{Ext}^i_R(\omega, F \otimes_R H^d_{\mathfrak{m}}(\omega)) \cong \operatorname{Ext}^i_R(\omega, F \otimes_R H^d_{\mathfrak{m}\widehat{R}}(\omega \otimes_R \widehat{R})) \cong \operatorname{Ext}^i_R(\omega, F \otimes_R E(R/\mathfrak{m})) = 0$ for any flat R-module F and any $i > 0$, the exact sequences (•) and (••) imply that $\operatorname{Ext}^1_R(\omega, \omega \otimes_R H^d_{\mathfrak{m}}(M)) = 0$, and for each $i > 1$, there is an isomorphism $\operatorname{Ext}^{i-1}_R(\omega, \omega \otimes_R H^d_{\mathfrak{m}}(K)) \cong \operatorname{Ext}^i_R(\omega, \omega \otimes_R H^d_{\mathfrak{m}}(M))$. Replacing M by K, we deduce $\operatorname{Ext}^2_R(\omega, \omega \otimes_R H^d_{\mathfrak{m}}(M)) = 0$. Now, an easy induction on i implies that $\operatorname{Ext}^i_R(\omega, \omega \otimes_R H^d_{\mathfrak{m}}(M)) = 0$ for each $i > 0$; hence the proof is finished. □

It should be noted that over a local ring $(R, \mathfrak{m})$ of Krull dimension d, the G-flat modules are strongly torsion free. In fact, if M is a G-flat R-module, then there exists an exact sequence $0 \to M \to F^0 \to F^1 \to \cdots$ of R-modules such that each F_i is flat, and for each $i \geq 1$, the module $K_i = \operatorname{Ker}(F_i \to F_{i+1})$ is G-flat. Let X be any R-module of finite flat dimension. Using [10, p. 84], we have $\operatorname{pd}_R X \leq d$, and hence $\operatorname{fd}_R X \leq d$. Now, we have $\operatorname{Tor}^R_1(X, M) \cong \operatorname{Tor}^R_2(X, K_1) \cong \cdots \cong \operatorname{Tor}^R_{d+1}(X, K_k) = 0$.

Proposition 4.13. *Let M be a strongly torsion free R-module such that $M \in \mathcal{G}_0(R)$. Then $H^d_{\mathfrak{m}}(M) \in \mathcal{G}_0(R)$.*

Proof. As $M \in \mathcal{G}_0(R)$, it follows from [5, Proposition 10.7.14 and Corollary 10.4.29] that M has finite G-flat dimension. Let $\operatorname{Gfd}_R M = n$. We proceed by induction on n. If $n = 0$, then M is G-flat, and then the result follows by the previous theorem. So, assume that $n > 0$ and that the result has been proved for all values smaller than n. Since $\operatorname{Gfd}_R M = n$, there is an exact sequence of R-modules $0 \to K \to F \to M \to 0$ such that $\operatorname{Gfd}_R K \leq n - 1$ and F is G-flat. It is easy to see that K is strongly torsion free. On the other hand, suppose that $x_1, \ldots, x_d \in \mathfrak{m}$ is a system of parameters of R. Since R is Cohen–Macaulay, $x_1, \ldots, x_d$ is an R-sequence. So in view of [9, Theorem 2] and the fact that M is strongly torsion free, for all $i < d$, we have $H^i_{\mathfrak{m}}(M) \cong H^i_{(x_1,\ldots,x_d)R}(M) \cong \varinjlim_{(\alpha_1,\ldots,\alpha_d)} \operatorname{Ext}^i_R(R/(x_1^{\alpha_1}, \ldots, x_d^{\alpha_d})R, M) \cong \varinjlim_{(\alpha_1,\ldots,\alpha_d)} \operatorname{Tor}^R_{d-i}(R/(x_1^{\alpha_1}, \ldots, x_d^{\alpha_d})R, M) = 0$.
So applying the functor $\Gamma_{\mathfrak{m}}(-)$ to the exact sequence above induces the following exact sequence:

$$0 \to H^d_{\mathfrak{m}}(K) \to H^d_{\mathfrak{m}}(F) \to H^d_{\mathfrak{m}}(M) \to 0.$$

By the previous theorem and by the induction hypothesis, we have $H^d_{\mathfrak{m}}(F), H^d_{\mathfrak{m}}(K) \in \mathcal{G}_0(R)$. Therefore $H^d_{\mathfrak{m}}(M) \in \mathcal{G}_0(R)$. □

Theorem 4.14. *Let M be a maximal Cohen–Macaulay R-module such that $M \in \mathcal{G}_0(R)$. Then $\operatorname{Hom}_R(M, \omega) \in \mathcal{J}_0(R)$.*

Proof. As M is maximal Cohen–Macaulay, by using [3, Theorem 2.8], it is strongly torsion free. Now, using Proposition 4.13, we have $H^d_{\mathfrak{m}}(M) \in \mathcal{G}_0(R)$, and hence by using [5, Proposition 10.4.17 and Corollary 10.4.29], the module $H^d_{\mathfrak{m}}(M)$ is of finite G-flat dimension. One can easily show that $\operatorname{Hom}_R(N, E(R/\mathfrak{m}))$ is G-injective for any G-flat module N. This fact implies that $\operatorname{Hom}_R(H^d_{\mathfrak{m}}(M), E(R/\mathfrak{m}))$ is of finite G-injective dimension, and so we have $\operatorname{Hom}_R(H^d_{\mathfrak{m}}(M), E(R/\mathfrak{m})) \in \mathcal{J}_0(R)$ by using [5, Proposition 10.4.23]. On the other hand, an application of the local duality theorem of local cohomology induces the following isomorphisms:

$$H^d_{\mathfrak{m}}(M) \cong H^d_{\mathfrak{m}}(M) \otimes_R \widehat{R} \cong H^d_{\mathfrak{m}\widehat{R}}(M \otimes_R \widehat{R}) \cong \operatorname{Hom}_{\widehat{R}}(\operatorname{Hom}_{\widehat{R}}(M \otimes_R \widehat{R}, \omega \otimes_R \widehat{R}), E(R/\mathfrak{m})).$$

Thus, we have the following isomorphism:

$$\begin{aligned}\operatorname{Hom}_R(M, \omega) \otimes_R \widehat{R} &\cong \operatorname{Hom}_{\widehat{R}}(M \otimes_R \widehat{R}, \omega \otimes_R \widehat{R}) \cong \operatorname{Hom}_{\widehat{R}}(H^d_{\mathfrak{m}}(M), E(R/\mathfrak{m}))\\ &\cong \operatorname{Hom}_{\widehat{R}}(H^d_{\mathfrak{m}}(M) \otimes_R \widehat{R}, E(R/\mathfrak{m})) \cong \operatorname{Hom}_R(H^d_{\mathfrak{m}}(M), E(R/\mathfrak{m})).\end{aligned}$$

Therefore, the preceding argument implies that $\operatorname{Hom}_R(M, \omega) \otimes_R \widehat{R} \in \mathcal{J}_0(R)$. Now, we prove that $\operatorname{Hom}_R(M, \omega) \in \mathcal{J}_0(R)$. Since $\operatorname{Hom}_R(M, \omega) \otimes_R \widehat{R} \in \mathcal{J}_0(R)$, we have the following natural isomorphisms:

$$\begin{aligned}\operatorname{Hom}_R(M, \omega) \otimes_R \widehat{R} &\cong \omega \otimes_R \operatorname{Hom}_R(\omega, \operatorname{Hom}_R(M, \omega) \otimes_R \widehat{R})\\ &\cong \omega \otimes_R \operatorname{Hom}_R(\omega, \operatorname{Hom}_R(M, \omega)) \otimes_R \widehat{R}.\end{aligned}$$

As $\widehat{R}$ is a faithfully flat R-module, the isomorphisms above imply a natural isomorphism $\mathrm{Hom}_R(M,\omega) \cong \omega \otimes_R \mathrm{Hom}_R(\omega, \mathrm{Hom}_R(M,\omega))$. On the other hand, for each $i > 0$, we also have the following isomorphisms:

$$0 = \mathrm{Ext}^i_R(\omega, \mathrm{Hom}_R(M,\omega) \otimes_R \widehat{R}) \cong \mathrm{Ext}^i_R(\omega, \mathrm{Hom}_R(M,\omega)) \otimes_R \widehat{R}.$$

Again, since $\widehat{R}$ is a faithfully flat R-module, we have $\mathrm{Ext}^i_R(\omega, \mathrm{Hom}_R(M,\omega)) = 0$. Lastly we should prove that for each $i > 0$, $\mathrm{Tor}^R_i(\omega, \mathrm{Hom}_R(\omega, \mathrm{Hom}_R(M,\omega))) = 0$. In view of [13, Proposition 3.2.9 and Corollary 3.2.10] and the fact that $\mathrm{Hom}_R(M,\omega) \otimes_R \widehat{R} \in \mathcal{J}_0(R)$, for each $i > 0$, we have the following isomorphisms:

$$\begin{aligned} 0 &= \mathrm{Tor}^R_i(\omega, \mathrm{Hom}_R(\omega, \mathrm{Hom}_R(M,\omega) \otimes_R \widehat{R})) \cong \mathrm{Tor}^R_i(\omega, \mathrm{Hom}_R(\omega, \mathrm{Hom}_R(M,\omega)) \otimes_R \widehat{R}) \\ &\cong \mathrm{Tor}^{\widehat{R}}_i(\omega \otimes_R \widehat{R}, \mathrm{Hom}_R(\omega, \mathrm{Hom}_R(M,\omega)) \otimes_R \widehat{R}) \cong \mathrm{Tor}^R_i(\omega, \mathrm{Hom}_R(\omega, \mathrm{Hom}_R(M,\omega))) \otimes_R \widehat{R}. \end{aligned}$$

The faithful flatness of $\widehat{R}$ implies $\mathrm{Tor}^R_i(\omega, \mathrm{Hom}_R(\omega, \mathrm{Hom}_R(M,\omega))) = 0$; hence the result follows. $\square$

Acknowledgments

I would like to thank the referee for his/her detailed reading of this paper and many thoughtful suggestions. This research was in part supported by a grant from the IPM (No. 85130029).

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